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# On the integrability of Friedmann–Robertson–Walker models with conformally coupled massive scalar fields

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**Abstract**
In this paper, we use a nonintegrability theorem by Morales and Ramis to analyse the integrability of Friedmann–Robertson–Walker cosmological models with a conformally coupled massive scalar field. We answer the long-standing question of whether these models with a vanishing cosmological constant and non-self-interacting scalar field are integrable: by applying Kovacic's algorithm to the normal variational equations, we prove analytically and rigorously that these equations and, consequently, the Hamiltonians are nonintegrable. We then address the models with a self-interacting massive scalar field and cosmological constant and show that, with the exception of a set of measure zero, the models are nonintegrable. For the spatially curved cases, we prove that there are no additional integrable cases other than those identified in the previous work based on the non-rigorous Painlevé analysis. In our study of the spatially flat model, we explicitly obtain a new possibly integrable case.

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## 1. Introduction

In recent years, the search for nonintegrability criteria for the Hamiltonian systems in the complex domain has acquired more relevance [1–5]. Such techniques are potentially of particular importance in cosmology because of controversies over both integrability, and the existence of chaos in cosmological models [6–16]. A part of the problem is that certain methods used to traditionally measure chaos in non-relativistic systems, such as the Lyapunov exponents, are no longer valid in general relativity where there is no absolute time coordinate.

In this work, we use a theorem by Morales and Ramis [1] which establishes a relation between two different concepts of integrability: the complete integrability of complex analytical Hamiltonian systems (given by Liouville's theorem) and the integrability of homogeneous linear ordinary differential equations in terms of Liouvillian functions in the complex plane. A Liouvillian function is a function which can be written as a combination of elementary functions, algebraic functions (solutions of polynomial equations), their indefinite integrals or exponentials of these integrals. Since we are working in the complex domain, this definition includes (but is not limited to) functions such as logarithms, trigonometric functions and their inverses.

In this paper, we use the conformal form of the FRW metric

$$\mathrm{d}s^2 = a^2(\eta)\left[\mathrm{d}\eta^2 - \frac{1}{1-kr^2}\mathrm{d}r^2 - r^2\,\mathrm{d}\Omega^2\right], \tag{1}$$

with $\eta$ being the conformal time, $a(\eta)$ the scale factor and $k = 0, \pm 1$ the curvature.

It is known [9, 12] that a suitable Hamiltonian which describes a cosmological model with this metric, and a conformally coupled scalar field, $\phi$, of mass $m$ is

$$H = \frac{1}{2}\left[-\left(p_a^2 + ka^2\right) + \left(p_\phi^2 + k\phi^2\right) + m^2a^2\phi^2 + \frac{\lambda\phi^4}{2} + \frac{\Lambda a^4}{2}\right], \tag{2}$$

with $\Lambda$ being the cosmological constant, $\lambda$ the self-coupling of $\phi$ and $p_a$ and $p_\phi$ the momenta conjugate to $a$ and $\phi$, respectively.

Since the cases of the free and self-interacting scalar field are best solved using different techniques, we treat these cases separately: in section 3 we study the massive scalar field with $\lambda = \Lambda = 0$, and in section 4 the general case. Before presenting our analysis, we describe briefly the Morales–Ramis theorem which is central to our work.

## 2. The Morales–Ramis Theorem

The Morales–Ramis Theorem (MRT) rigorously provides necessary conditions for the integrability of a Hamiltonian system and so sufficient conditions for nonintegrability. The theorem is based on the analysis of the variational equations (in particular, the normal variational equation, or NVE) for the perturbations of a non-equilibrium particular solution. The basic idea is that if the flow of the Hamiltonian system has a regular behaviour (is integrable), then the linearized flow along a particular integral curve given by the NVE must also be regular (integrable). Conversely if the linearized flow is nonintegrable the system as a whole will be nonintegrable.

The Hamiltonian system, $X_H$, of dimension $n$ is called integrable if there exist $n$ independent constants of the motion in involution. By considering the differential Galois group of the NVE, the theorem of Morales–Ramis links this concept of integrability to an apparently different concept of integrability—the existence of Liouvillian solutions of the NVE of $X_H$. The theorem may be stated as follows.

**Theorem 1.** *If there are n first integrals of $X_H$ that are independent and in involution, then the identity component of the Galois group of the NVE is Abelian.*

It is known that [17] for an ODE to admit a Liouvillian solution, the identity component of its Galois group must be soluble. Hence, if the solutions are not Liouvillian, the identity component of the Galois group is not soluble and, therefore, non-Abelian.

Our strategy will therefore be to

(1) select a particular solution;

(2) obtain the variational equations and the NVE;
(3) check if the NVE has any Liouvillian solutions.

There are many ways of performing the third step, with the efficiency dependent on the Hamiltonian (and NVEs) under study. Because of this we find it useful to apply different methods, such as Kovacic's algorithm [18], or a theorem by Kimura [19], for specific cases of the Hamiltonian.

## 3. The noninteracting scalar field

The dynamics of the FRW model with a massive noninteracting scalar field has been discussed and studied before using numerical methods [6–10]. Though the results are strongly suggestive of nonintegrability, by their nature these numerical studies do not constitute a proof.

Several papers have attacked the problem of the integrability of the dynamical system analytically by studying the integrability of the corresponding Hamiltonian [6–10]:

$$H = \tfrac{1}{2}\left[\left(p_\phi^2 + k\phi^2\right) - \left(p_a^2 + ka^2\right) + m^2a^2\phi^2\right]. \tag{3}$$

We note that, due to the Hamiltonian constraint of General Relativity we have that, for vacuum, $H = 0$. Using monodromy groups, Ziglin [11] concluded that the Hamiltonian (3) is nonintegrable on the interval $0 < H < 1/2m^2$, which excludes the particularly interesting case for General Relativity, $H = 0$. The intention of this part of the paper is to fill that gap.

### *3.1. The case* $k \neq 0$

We use the theorem of Morales–Ramis with (3). We choose as our set of non-equilibrium particular solutions the invariant plane $p_a = a = 0$. The NVEs relative to this plane are

$$\frac{\mathrm{d}^2\delta\, a}{\mathrm{d}t^2} = (-k + m^2\phi^2)\delta a. \tag{4}$$

Changing the independent variable to $\phi$ and renaming $\delta a = y$, we obtain the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\phi^2} + \frac{1}{\phi}\frac{\mathrm{d}y}{\mathrm{d}\phi} + \left(\frac{m^2}{k} - \frac{1}{\phi^2}\right) y = 0. \tag{5}$$

This equation is a second-order, linear and homogeneous ODE with coefficients which are rational functions of $\phi$. We now need to determine whether this ODE has any Liouvillian solutions. For this case it is convenient to apply Kovacic's algorithm which, though complex to write down, is, as we shall see, straightforward to apply to the problem in hand. Since the various versions of Kovacic's algorithm in the literature [1, 16, 18, 20] have slight differences in the presentation and conventions, we include the version of the algorithm used by us in the appendix. We show how the algorithm quickly determines that the Hamiltonian is nonintegrable for $k \neq 0$.

When $k = 0$ the analysis based on the invariant planes $a = p_a = 0$ and $\phi = p_\phi = 0$ is inconclusive. However, because the potential is homogeneous in this case, there exist particular nonsingular solutions which lie outside these planes and can be used as a basis for the analysis. Fortunately, the case of homogeneous potentials has been exhaustively studied by Yoshida [5] and Morales–Ramis [1] and so we can simply apply those results.

Using the well-known transformations (see the appendix), we transform (5) into the reduced invariant form:

$$\xi'' = \left(\frac{3k - 4m^2\phi^2}{4k\phi^2}\right)\xi. \tag{6}$$

**Lemma 1.** *Equation (5) has no Liouvillian solutions when $k \neq 0$ and $m \neq 0$.*

**Proof.** By the application of Kovacic's algorithm to equation (6).

*Step 1.* $g(\phi)$ has one finite pole, at $\phi = 0$, of order 2 and the pole at infinity, of order 4 (since, by assumption, we are treating the massive case, $m \neq 0$). This implies that $m^+ = 4$ and $\gamma = \gamma_2 = 1$. Since the pole at $\phi = 0$ belongs to $\Gamma_2$, we calculate the Laurent series (when $k \neq 0$) as

$$g_0 = \frac{3}{4}\phi^{-2} - \frac{m^2}{k}.$$

Hence $\alpha_0 = \frac{3}{4}$ and $\beta_0 = 0$. Thus we have $L = \{1\}$.

*Step 2.* Because $L = \{1\}$ the unique value for $n$ is $n = 1$. Through the items 2.3 and 2.5, we calculate the sets $E_c$. From 2.3 we have that $E_0 = \left\{\frac{3}{2}, -\frac{1}{2}\right\}$. In item 2.5.2, we need to expand $g$ around $\phi = \infty$. Doing this we obtain

$$g_\infty = \frac{3}{4}\phi^{-2} - \frac{m^2}{k} \quad \Longrightarrow \quad E_\infty = \{1\}.$$

Summarizing,

$$E_0 = \left\{\frac{3}{2}, -\frac{1}{2}\right\} \qquad \text{and} \qquad E_\infty = \{1\}.$$

*Step 3.* In this step, we need to calculate $\prod_{c\in\Gamma} E_c$, using the sets determined in the previous step. We obtain the set of sets given by

$$\prod_{c\in\Gamma} E_c = \left\{\left\{\frac{3}{2}, 1\right\}, \left\{-\frac{1}{2}, 1\right\}\right\}.$$

From 3.1(i) we calculate the values of $d(\mathbf{e})$ as $d = -\frac{3}{2}$ and $d = \frac{1}{2}$, respectively. Since neither of these values satisfies 3.1(i) and there are no other values of $n$ in $L$, the Galois group of (6) is $SL(2, \mathbb{C})$, equation (6) is nonintegrable in terms of Liouvillian functions, and therefore the system represented by the Hamiltonian (3) is also nonintegrable when $k \neq 0$. This completes the proof. □

### 3.2. *The case $k = 0$*

When $k = 0$, the Hamiltonian (3) assumes the form

$$H = \tfrac{1}{2}\left(p_\phi^2 - p_a^2 + m^2 a^2 \phi^2\right), \tag{7}$$

with a manifestly homogeneous potential. The Hamiltonians with homogeneous potentials have generated interest in the literature, stimulated by the results of Yoshida [5], refined and generalized by Morales–Ramis [1], which give the easily used criteria for deciding when such Hamiltonians may be integrable. We shall refer to these results as the Morales–Ramis–Yoshida (MRY) theorem. We note that the Hamiltonian (2) with $k = 0$ (or essentially equivalent Hamiltonians) have been studied in detail recently [21–24].

Explicitly, the MRY theorem treats Hamiltonians of the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \ldots, q_n), \tag{8}$$

where $V$ is a homogeneous potential of degree $g$ (i.e., $V(a\overrightarrow{Q}) = a^g V(\overrightarrow{Q})$, for $a$ constant). Let $c$ be a solution of the equation $c = \overrightarrow{V}'(c)$ and $Y_i$ (the Yoshida coefficients) the eigenvalues of the matrix $V''(c)$ (note that one of these eigenvalues is trivial, in that it corresponds to the tangential variational equation, and has the value $g-1$). Then we may state the MRY theorem as follows.

**Theorem 2** (Morales–Ramis–Yoshida). *If the Hamiltonian system of the form (8) is completely integrable (with holomorphic or meromorphic first integrals), then each pair* $(g, Y_i)$ *belongs to one of the following list (where we do not consider the trivial case* $g = 0$*):*

$$\begin{array}{ll}
(1) & (g, p + p(p-1)g/2); \\
(2) & (2, \textit{arbitrary complex number}); \\
(3) & (-2, \textit{arbitrary complex number}); \\
(4) & \left(-5, \frac{49}{40} - \frac{1}{40}\left(\frac{10}{3} + 10p\right)^2\right); \\
(5) & \left(-5, \frac{49}{40} - \frac{1}{40}(4 + 10p)^2\right); \\
(6) & \left(-4, \frac{9}{8} - \frac{1}{8}\left(\frac{4}{3} + 4p\right)^2\right); \\
(7) & \left(-3, \frac{25}{24} - \frac{1}{24}(2 + 6p)^2\right); \\
(8) & \left(-3, \frac{25}{24} - \frac{1}{24}\left(\frac{3}{2} + 6p\right)^2\right); \\
(9) & \left(-3, \frac{25}{24} - \frac{1}{24}\left(\frac{6}{5} + 6p\right)^2\right); \\
(10) & \left(-3, \frac{25}{24} - \frac{1}{24}\left(\frac{12}{5} + 6p\right)^2\right); \\
(11) & \left(3, -\frac{1}{24} + \frac{1}{24}(2 + 6p)^2\right); \\
(12) & \left(3, -\frac{1}{24} + \frac{1}{24}\left(\frac{3}{2} + 6p\right)^2\right); \\
(13) & \left(3, -\frac{1}{24} + \frac{1}{24}\left(\frac{6}{5} + 6p\right)^2\right); \\
(14) & \left(3, -\frac{1}{24} + \frac{1}{24}\left(\frac{12}{5} + 6p\right)^2\right); \\
(15) & \left(4, -\frac{1}{8} + \frac{1}{8}\left(\frac{4}{3} + 4p\right)^2\right); \\
(16) & \left(5, -\frac{9}{40} + \frac{1}{40}\left(\frac{10}{3} + 10p\right)^2\right); \\
(17) & \left(5, -\frac{9}{40} + \frac{1}{40}(4 + 10p)^2\right); \\
(18) & \left(g, \frac{1}{2}\left[\frac{g-1}{g} + p(p+1)g\right]\right);
\end{array} \tag{9}$$

*where p is an arbitrary integer.*

**Lemma 2.** *When* $k = 0$ *the only first integral of the Hamiltonian system (3) is the Hamiltonian, and the system is therefore nonintegrable.*

**Proof.** By the application of the MRY theorem. For the system represented by (7) the only nontrivial Yoshida coefficient is $Y_1 = -1$. For $g = 4$, the only possibilities for satisfying theorem 3 are (1), (15) and (18). For all these cases there are no integer values of $p$ which solve $Y_1 = -1$, and we conclude that the system represented by (7) is nonintegrable. This completes the proof of lemma 2.

We can now enunciate the following theorem. □

**Theorem 3.** *The Friedmann–Robertson–Walker model with a conformally coupled massive scalar field represented by the Hamiltonian (2) with* $H = 0$ *is not completely integrable.*

**Proof.** By lemma 1, there are no Liouvillian solutions of the NVE for the plane $a = p_a = 0$. This implies that the identity component of the differential Galois group of the NVE is not soluble and therefore non-Abelian. Using theorem 1, we have that the only first integral of the Hamiltonian system is the Hamiltonian itself, and that the system is not completely integrable in this case. By lemma 2, the Hamiltonian system is also not completely integrable when $k = 0$. Therefore, the Hamiltonian system represented by (2) with $H = 0$ is nonintegrable for all values of $k$. □

## 4. The general case

In this section, we study FRW spacetimes with a self-interacting scalar field, $\phi$, of mass $m$, conformally coupled to gravity, and with a cosmological constant, $\Lambda$. Since the NVEs in this case are more complex, we demonstrate the nonintegrability of these equations by transforming them into hypergeometric equations, and applying the results of Kimura [19] for these equations to establish their generic nonintegrability.

The Hamiltonian (2) (or ostensibly equivalent ones) has been extensively studied in the literature [9, 12, 21] but, in our opinion, for reasons described below, these works are either incomplete or non-rigorous. Besides the initial numerical studies by Blanco *et al* [9], early investigations of the integrability of (2) can be traced back to Lakshmanan and Sahadevan [28] who studied a variety of Hamiltonians for coupled nonlinear anharmonic oscillators using the ARS algorithm [25–27] and determined explicitly many first integrals other than the Hamiltonians. Strictly speaking, they treated positive-definite Hamiltonians, but the simple canonical transformation

$$a \to \mathrm{i}a, \qquad p \to -\mathrm{i}p \tag{10}$$

brings (2) into the positive definite form. Their results are extensive and include (2) as a particular case. However, as we have mentioned, their use of the Painlevé analysis and the ARS algorithm leaves some doubt in the air, in particular as to the integrability of the other cases.

Though there is a strong connection between integrability and the Painlevé property, and the latter has been remarkably successful in indicating possibly integrable cases, it is worth noting that the lack of the Painlevé property is not a rigorous obstruction to integrability [29]. Additionally, the ARS algorithm is not a foolproof method for determining whether a system possesses the Painlevé property, and its application can lead to false conclusions [13–15], particularly when applied to determining the nonintegrability of a dynamical system.

Helmi and Vucetich [12] applied the ARS algorithm explicitly to the Hamiltonian (2) transformed by (10) for both $k = 0$ and $k \neq 0$. Equating the parameters used in [12, 28], we find that their results are essentially the same. When $k \neq 0$, they identified the possibly integrable cases

$$\Lambda = \lambda = -m^2/3, \qquad \Lambda = \lambda = -m^2, \tag{11}$$

while, for $k = 0$, they gave the additional possibilities

$$\Lambda = 8\lambda = -8m^2/3, \qquad \Lambda = 16\lambda = -8m^2/3. \tag{12}$$

That these cases are indeed integrable was confirmed by finding explicitly the corresponding first integrals, as was done in [28]. Once again the use of the Painlevé analysis means that nothing definitive can be said about the integrability for other values of the parameters. In fact, as we shall see, by applying the MRT we identify other combinations of the parameters when $k = 0$ which *may* produce integrable Hamiltonians. When $k \neq 0$ we can, however, show that the model must be nonintegrable for all other combinations of parameters.

More recently, the Hamiltonian (2) with $k = 0$ has been studied by Maciejewski and Szydłowski [21, 22] using rigorous methods developed by Ziglin [2] and Morales–Ramis

[1]. In this case, the potential is homogenous, which simplifies the analysis, since the case of homogenous potentials was studied by Yoshida [5] and later extended by Morales–Ramis using the differential Galois group theory. Using these results they conclude (rigorously) that the system is generically nonintegrable. They give conditions on the parameters for possibly integrable cases, and find one set of solutions with $\lambda = -m^2$. By symmetry, a similar set with $\Lambda = -m^2$ obviously exists. However, no other possibly integrable cases are identified. In fact, since the general problem involves solving a system of three diophantine equations in two parameters, it is not even evident that other solutions exist. In the same paper, it is implied that the case $k \neq 0$ is also nonintegrable but no proof or analysis supporting this conclusion is presented by the authors.

Later, the authors [31] analysed the Hamiltonian (2) using the MRT but omitted the analysis of a third invariant plane which places further restrictions on possibly integrable cases. Most recently, Boucher and Weil [32] study (2) using the 'higher variational theorem' of Morales–Ramis–Simó [33]. They conjecture that, apart from the cases cited by Helmi and Vucetich, there are no other integrable cases, and provide strong evidence for this result. Here we prove this conjecture.

In summary, initial studies of (2) have used the Painlevé analysis which, though strongly related to integrability, is not always reliable. Later works have used rigorous methods but are incomplete: the cases $k \neq 0$ (and the consequent inhomogeneous potential) have not been covered, and when $k = 0$ all possibly integrable cases have not been explicitly identified.

Here we use the MRT to (i) rigorously prove that the Hamiltonian (2) is generically nonintegrable when $k \neq 0$; (ii) together with the results of Boucher and Weil, prove that the only integrable cases when $k \neq 0$ are those identified by Helmi and Vucetich, proving Boucher and Weil's conjecture; (iii) identify, in the case $k = 0$, a new explicit combination of the parameters for which the model is possibly integrable (in other words, nonintegrability is uncertain).

We apply different theorems when $k \neq 0$ and $k = 0$ since, as was the case with the free scalar field, the analysis of the invariant planes is not conclusive when $k = 0$. When $k \neq 0$ the NVE can be transformed into a hypergeometric equation. The problem of when the hypergeometric equation has a solution in terms of Liouvillian functions was solved by Kimura [19] and we can apply his results. Again the case $k = 0$ results in a homogeneous potential, and the integrability of these Hamiltonians can be studied using the MRY theorem.

## 5. Model with $k \neq 0$

To apply the MRT we first need a set of non-equilibrium particular solutions about which we will perform the perturbations to obtain the NVEs. Two obvious invariant planes are $a = p_a = 0$ and $\phi = p_\phi = 0$. To identify a third invariant plane, we make a canonical change of variables in which the new coordinates $(Q_1, Q_2)$ and the new momenta $(P_1, P_2)$ are linear combinations of $(a, \phi)$ and $(P_a, P_\phi)$, respectively, and impose that $(\dot{Q}_1 = 0, \dot{P}_1 = 0)$. With these conditions a new invariant plane is $(Q_1 = 0, P_1 = 0)$, where

$$
\begin{aligned}
Q_1 &= \sqrt{\Lambda + m^2}a + \sqrt{-(\lambda + m^2)}\phi, \\
Q_2 &= \frac{\sqrt{\Lambda\lambda - m^4}}{\sqrt{2k}(\Lambda + 2m^2 + \lambda)}\left(\sqrt{\lambda + m^2}a + \sqrt{\Lambda + m^2}\phi\right), \\
P_1 &= \frac{1}{\Lambda + 2m^2 + \lambda}\left(\sqrt{\Lambda + m^2}P_a - \sqrt{-(\lambda - m^2)}P_\phi\right), \\
P_2 &= \sqrt{\frac{2k}{\Lambda\lambda - m^4}}\left(\sqrt{\lambda + m^2}P_a + \sqrt{-(\Lambda + m^2)}P_\phi\right).
\end{aligned}
$$

We note that the transformation is singular when $\lambda = \Lambda = -m^2$ and so this case should strictly be treated separately but, since this case is already known to be integrable, it is not necessary to pursue the analysis further.

Turning to the calculation of the NVEs, for the invariant plane $p_a = a = 0$ the NVE is

$$\frac{\mathrm{d}^2\delta a}{\mathrm{d}t^2} = (-k + m^2\phi^2)\delta a. \tag{13}$$

Renaming $\delta a = x$, changing the independent variable to $\phi$, and using the Hamiltonian to evaluate the derivatives of $\phi$, we obtain

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\phi^2} + \frac{2(k + \lambda\phi^2)}{\phi(2k + \lambda\phi^2)}\frac{\mathrm{d}x}{\mathrm{d}\phi} + \frac{2(-k + m^2\phi^2)}{\phi^2(2k + \lambda\phi^2)}x = 0. \tag{14}$$

We note that this is a linear homogeneous ODE whose coefficients are rational functions, and which has three finite poles: two simple and one double. Since $\lambda \neq 0$, we can transform (14) to a Heun equation by the sequence of S-Homotopic and Möbius transformations $x = y\phi$ and $\phi = \sqrt{\frac{-2k}{\lambda}}z$, respectively, after which we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \frac{3 - 4z^2}{z(z^2 - 1)}\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{2(m^2 + \lambda)y}{\lambda(z^2 - 1)} = 0. \tag{15}$$

Applying the recent work by Maier [34], we transform (15) into a hypergeometric equation with the Heun-to-hypergeometric reduction $z = 1 - \Phi^2$ and obtain

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\Phi^2} + \frac{-1 + 5\Phi}{2\Phi(\Phi - 1)}\frac{\mathrm{d}y}{\mathrm{d}\Phi} + \frac{(m^2 + \lambda)y}{2\lambda\Phi(\Phi - 1)} = 0. \tag{16}$$

We compare (16) with the general form of the hypergeometric equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{[\gamma - (\alpha + \beta + 1)x]}{x(1 - x)}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\alpha\beta y}{x(1 - x)} = 0, \tag{17}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are parameters, and identify $\gamma = 1/2$, and the two possible pairs of values for $\alpha$ and $\beta$:

$$\beta_1 = \frac{3 + \sqrt{1 - 8m^2/\lambda}}{4} \qquad \text{and} \qquad \alpha_1 = \frac{3 - \sqrt{1 - 8m^2/\lambda}}{4},$$
$$\beta_2 = \frac{3 - \sqrt{1 - 8m^2/\lambda}}{4} \qquad \text{and} \qquad \alpha_2 = \frac{3 + \sqrt{1 - 8m^2/\lambda}}{4}.$$

We now turn to Kimura's theorem [1, 19] which tells us for which combinations of the parameters $\{\alpha, \beta, \gamma\}$ equation (17) has a Liouvillian solution.

**Theorem 4** (Kimura)**.** *For the hypergeometric equation (17), define the exponent differences* $\hat{\lambda} = 1 - \gamma$, $\hat{\mu} = \alpha - \beta$ *and* $\hat{\nu} = \gamma - \alpha - \beta$. *Then the identity component of the Galois group of the equation is soluble if, and only if, either*

*(A) at least one of* $\hat{\lambda} + \hat{\mu} + \hat{\nu}$, $-\hat{\lambda} + \hat{\mu} + \hat{\nu}$, $\hat{\lambda} - \hat{\mu} + \hat{\nu}$ *and* $\hat{\lambda} + \hat{\mu} - \hat{\nu}$ *is an odd integer, or*

*(B) each of the quantities* $\pm\hat{\lambda}$, $\pm\hat{\mu}$ *and* $\pm\hat{\nu}$ *belong (in an arbitrary order) to one of the following 15 families,*

| 1 | $1/2+l$ | $1/2+m$ | Arbitrary complex number | |
|---|---|---|---|---|
| 2 | $1/2+l$ | $1/3+m$ | $1/3+q$ | |
| 3 | $2/3+l$ | $1/3+m$ | $1/3+q$ | $l+m+q$ even |
| 4 | $1/2+l$ | $1/3+m$ | $1/4+q$ | |
| 5 | $2/3+l$ | $1/4+m$ | $1/4+q$ | $l+m+q$ even |
| 6 | $1/2+l$ | $1/3+m$ | $1/5+q$ | |
| 7 | $2/5+l$ | $1/3+m$ | $1/3+q$ | $l+m+q$ even |
| 8 | $2/3+l$ | $1/5+m$ | $1/5+q$ | $l+m+q$ even |
| 9 | $1/2+l$ | $2/5+m$ | $1/5+q$ | $l+m+q$ even |
| 10 | $3/5+l$ | $1/3+m$ | $1/5+q$ | $l+m+q$ even |
| 11 | $2/5+l$ | $2/5+m$ | $2/5+q$ | $l+m+q$ even |
| 12 | $2/3+l$ | $1/3+m$ | $1/5+q$ | $l+m+q$ even |
| 13 | $4/5+l$ | $1/5+m$ | $1/5+q$ | $l+m+q$ even |
| 14 | $1/2+l$ | $2/5+m$ | $1/3+q$ | $l+m+q$ even |
| 15 | $3/5+l$ | $2/5+m$ | $1/3+q$ | $l+m+q$ even |

*where $l$, $m$ and $q$ are integers.*

Writing $c \equiv \frac{1}{2}\sqrt{1-8m^2/\lambda}$, we have two possible sets of $\{\hat{\lambda}, \hat{\mu}, \hat{\nu}\}$ for the transformed NVE (16), namely,

$$\hat{\lambda} = \tfrac{1}{2}, \qquad \hat{\mu}_1 = -c, \qquad \hat{\nu}_1 = \gamma - \alpha_1 - \beta_1 = -1,$$
$$\hat{\lambda} = \tfrac{1}{2}, \qquad \hat{\mu}_2 = +c, \qquad \hat{\nu}_2 = \gamma - \alpha_2 - \beta_2 = -1.$$

We discover that (16) satisfies case (A) if $c = (2K+1)/2$, $K \in \mathbb{Z}$. Turning now to case (B), of the 15 possibilities in the table, we find that only the first is consistent with our values of $\hat{\lambda}$, $\hat{\mu}$ and $\hat{\nu}$. However, the restriction obtained on the value of $c$ is the same as that obtained in the case (A). We therefore conclude that (14) has a Liouvillian solution if, and only if,

$$\sqrt{1-8m^2/\lambda} = 2M+1, \qquad M \in \mathbb{Z}. \tag{18}$$

We can exclude the possibility $M = 0$ since that corresponds to either $m = 0$ or $\lambda = \infty$.

Because of the manifest symmetry between $a$ and $\phi$ in the Hamiltonian, the analysis for the invariant plane $p_\phi = \phi = 0$ is almost identical and we determine that the NVE for this plane has Liouvillian solutions if, and only if,

$$\sqrt{1-8m^2/\Lambda} = 2N+1, \qquad N \in \mathbb{Z}. \tag{19}$$

Again we exclude $N = 0$ since then either $m = 0$ or $\Lambda = \infty$.

For the third invariant plane, we calculate the NVE

$$\frac{\mathrm{d}^2X}{\mathrm{d}Q^2} + \frac{2Q_2^2-1}{Q_2(Q_2-1)(Q_2+1)}\frac{\mathrm{d}X}{\mathrm{d}Q_2} - \frac{\left[4Q_2^2m^2(\Lambda+\lambda) + \Lambda\lambda\left(6Q_2^2-1\right) + m^4\left(2Q_2^2+1\right)\right]X}{Q_2^2(Q_2-1)(Q_2+1)(\Lambda\lambda-m^4)} = 0. \tag{20}$$

This equation can also be transformed into a hypergeometric equation. Using the same transformations as before ($X = Q_2Y$, followed by $\Phi = 1 - Q_2^2$), we have

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\Phi^2} + \frac{-5\Phi+1}{2\Phi(1-\Phi)}\frac{\mathrm{d}Y}{\mathrm{d}\Phi} + \frac{(\Lambda+m^2)(\lambda+m^2)Y}{\Phi(1-\Phi)(\Lambda\lambda-m^4)} = 0. \tag{21}$$

Applying Kimura's theorem to equation (21), we find that it is integrable if, and only if,

$$\sqrt{\frac{25\Lambda\lambda + 16m^2(\lambda+\Lambda) + 7m^4}{\Lambda\lambda - m^4}} = 2P+1, \qquad P \in \mathbb{Z}. \tag{22}$$

For the Hamiltonian to be integrable, it is necessary that (18), (19) and (22) hold simultaneously. Substituting (18) and (19) into (22), we find that $M$, $N$ and $P$ must be integers which satisfy

$$\sqrt{\frac{32N^2+32N-100-M(M+1)(7N^2+7N-32)}{M(M+1)N(N+1)-4}}=2P+1,$$

subject to $M \neq 0$ and $N \neq 0$. This rather daunting equation simplifies a little with the substitutions $M=\mu-\frac{1}{2}$, $N=\nu-\frac{1}{2}$:

$$\sqrt{-\frac{1863-540\mu^2-540\nu^2+112\nu^2\mu^2}{-63-4\mu^2-4\nu^2+16\nu^2\mu^2}}=2P+1, \qquad \mu\neq\frac{1}{2}, \quad \nu\neq\frac{1}{2}. \tag{23}$$

Since (23) is symmetric with respect to $\pm\mu \leftrightarrow \pm\nu$, we only need to study the quadrant $\mu \geqslant 0$, $\nu \geqslant 0$. In fact, without loss of generality, we need only study the octant where $\mu \geqslant \nu$. Now for an integer solution for $P$ to exist

$$\frac{1863-540\mu^2-540\nu^2+112\nu^2\mu^2}{63+4\mu^2+4\nu^2-16\nu^2\mu^2}\geqslant 0. \tag{24}$$

Substituting $\mu=\nu=0$ this is true in the neighbourhood of the origin. By analysing separately the curves along which the numerator and denominator change sign, and remembering that $\nu=\frac{1}{2}$ does not interest us, it is not difficult to establish that the only half-integer values of $\nu$ in the first octant which need to be checked are $\nu=\left\{\frac{3}{2},\frac{5}{2}\right\}$. For $\nu=\frac{3}{2}$ ($N=1$) we have that $\Lambda=-m^2$, while the only restriction on $\lambda$ is

$$\frac{8m^2}{\lambda}=1-4\mu^2$$

(with $\mu$ a half-integer). However, Boucher and Weil have shown that, when $\Lambda=-m^2$, the cases $\lambda\neq\Lambda$ (and so $\mu\neq\nu$) are nonintegrable.

When $\nu=\frac{5}{2}$, the only value of $\mu\geqslant\nu$ for which (24) is positive is $\mu=\frac{5}{2}$. This pair corresponds to $\lambda=\Lambda=-m^2/3$, which is known to be integrable.

Finally, putting together all the pieces we determine that, for $k\neq 0$, the only combinations of $\lambda$, $\Lambda$ and $m$ for which the Hamiltonian (2) is integrable are

$$\Lambda=\lambda=-m^2/3 \qquad \text{and} \qquad \Lambda=\lambda=-m^2, \tag{25}$$

exactly those found in [12], with the first integrals other than the Hamiltonian being identified using [28]. Since both these papers used the Painlevé analysis, nothing definitive could be said about the integrability of other cases. From our analysis using the MRT, we are able to close the cases left open by Boucher and Weil and state theorem 5

**Theorem 5.** *The Hamiltonian (2) with $k\neq 0$ is nonintegrable except when $\Lambda=\lambda=-m^2/3$ or $\Lambda=\lambda=-m^2$.*

demonstrating Boucher and Weil's conjecture.

## 6. Models with $k = 0$

As was previously mentioned, when $k=0$ the potential in the Hamiltonian (2) becomes homogeneous of degree 4, so we can apply the MRY theorem.

For the Hamiltonian (2) with $k=0$, the only relevant cases are (1), (15) and (18). The Yoshida coefficients are

$$\{Y_i\}=\left\{\frac{-m^2}{\lambda},\frac{-m^2}{\Lambda},\frac{m^4+2m^2(\Lambda+\lambda)+3\lambda\Lambda}{\lambda\Lambda-m^4}\right\}. \tag{26}$$

We may write

$$Y_3 = \frac{Y_1 Y_2 - 2(Y_1 + Y_2) + 3}{1 - Y_1 Y_2}$$

and so there are, in general, only two free parameters. Since each $Y_i$ has to satisfy one of the conditions (1), (15) and (18), the problem of identifying possibly integrable cases involves solving an overdetermined system of three diophantine equations in two parameters, and is not even evident that solutions exist.

However, solutions do exist: Helmi and Vucetich [12] identified the integrable cases

$$\begin{aligned} &\lambda = \Lambda = -m^2, &&\lambda = \Lambda = -m^2/3, \\ &\lambda = \Lambda/8 = -m^2/3, &&\lambda = \Lambda/16 = -m^2/6. \end{aligned} \tag{27}$$

Though not explicitly stated, the last two have obvious analogues given by the symmetry $\lambda \leftrightarrow \Lambda$. Maciejewski and Szydłowski [21, 22] noted that if $\lambda = -m^2$ ($Y_1 = 1$) then automatically $Y_3 = 1$ and $Y_2$ can be given by any value which satisfies any one of (1), (15) or (18) of the table above, with $p$ prescribed freely. Again, the obvious symmetry $\lambda \leftrightarrow \Lambda$ provides an equivalent set with $Y_2 = 1$. One of these solutions coincides with the first possibility in (27).

We have identified one further possibly integrable case which does not belong to either of the classes of Helmi and Vucetich, or Maciejewski *et al*. For this solution

$$\lambda = \frac{-m^2}{136}, \qquad \Lambda = \frac{-8m^2}{3}.$$

The symmetry $\lambda \leftrightarrow \Lambda$ again produces another case.

## 7. Conclusion

We have studied the integrability of the Hamiltonian which describes (in general) a FRW universe with a conformally coupled scalar field and a cosmological constant. Our analysis is separated into two parts, which use different techniques, but both of which rely on the Morales–Ramis theorem to prove nonintegrability.

In the special case of a massive, non-self-interacting conformally coupled scalar field with zero cosmological constant, we were able to show, by the use of Kovacic's algorithm, that the normal variational equations associated with the Hamiltonian subject to the energy constraint are never solvable in terms of Liouvillian functions, thereby establishing nonintegrability. This is compatible with the results from numerical analysis based on Poincaré sections [6] which indicate that the behaviour of the system is mathematically chaotic.

For the general case of a massive self-interacting scalar and a (possibly vanishing) cosmological constant, we have rigorously proven that when the curvature $k \neq 0$ the model is not integrable, except for two cases, proving a conjecture of Boucher and Weil. When $k = 0$ we have explicitly identified a new combination of the parameters for which the Hamiltonian is possibly integrable.

*Note added.* While this paper was being considered for publication, the authors became aware of the similar work by Maciejewski *et al* [36] who obtained essentially the same results as the authors'.

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## Appendix. Kovacic's algorithm

Kovacic's algorithm provides a procedure for computing the Liouvillian solutions of a homogeneous linear second-order differential equation. If the algorithm terminates negatively, we can conclude that no such solutions exist.

Let $\mathbb{C}(x)$ be the field of rational complex functions (the ratios of polynomials in $x$ with complex coefficients). It is well known that by using the change of dependent variable

$$y = \xi \exp\left(\frac{1}{2}\int b\,\mathrm{d}x\right), \tag{A.1}$$

the second-order homogeneous linear ODE

$$y'' + b(x)y' + c(x)y = 0$$

can be transformed to the so-called reduced invariant form

$$\xi'' - g\xi = 0, \tag{A.2}$$

where

$$g(x) = \tfrac{1}{2}b'(x) + \tfrac{1}{4}b(x)^2 - c(x). \tag{A.3}$$

Note that, if $b(x)$ and $c(x) \in \mathbb{C}(x)$ then $g(x) \in \mathbb{C}(x)$.

Moreover, using a further change of variables $v = \xi'/\xi$, equation (A.2) is transformed into the Riccati equation

$$v' + v^2 = g. \tag{A.4}$$

Now equation (A.2) has a Liouvillian solution if and only if equation (A.4) has an algebraic solution, that is $v$ solves a polynomial equation $f(v) = 0$, where the degree of $f$ (the minimal polynomial) in $v$ belongs to the set $L_{\max} = \{1, 2, 4, 6, 12\}$.

Kovacic's algorithm can be divided into three main steps: the first step is the determination of the subset of $L$ relevant for the linear ODE under consideration; the other two steps are devoted, respectively, to determining the existence of the minimal polynomial, and its construction. If the algorithm does not terminate successfully (i.e., equation (A.4) has no algebraic solution), then equation (A.2) has no solution in terms of Liouvillian functions.

In the version used of the algorithm we essentially follow [1, 16, 20, 35]. Let

$$g = g(x) = \frac{s(x)}{t(x)}, \tag{A.5}$$

with $s(x)$, $t(x)$ being relatively prime polynomials and $t(x)$ monic. Define the function $h$ on the set $L_{\max} = \{1, 2, 4, 6, 12\}$ by $h(1) = 1$, $h(2) = 4$, $h(4) = h(6) = h(12) = 12$.

*Step 1* (determination of possible orders of the minimal polynomial).

If $t(x) = 1$ then set $m = 0$, else factorize $t(x)$ into monic relatively prime polynomials

$$t(x) = t_1(x)t_2^2(x)\cdots t_m^m(x),$$

where $t_i$ have no multiple roots and $t_m \neq 1$.

Then,

(1.1) Let $\Gamma'$ be the set of roots of $t(x)$ (i.e., the singular points in the finite complex plane) and let $\Gamma = \Gamma' \cup \infty$ be the set of singular points.

Then the order of a singular point $c \in \Gamma'$ is, as usual, $o(c) = i$ if $c$ is a root of multiplicity $i$ of $t_i$. The order at infinity is defined by $o(\infty) = \max(0, 4+\deg(s)-\deg(t))$. We call $m^+ = \max(m, o(\infty))$.

For $0 \leqslant i \leqslant m^+$, denote by $\Gamma_i = \{c \in \Gamma \mid o(c) = i\}$ the subset of all elements of order $i$.

(1.2) If $m^+ \geqslant 2$, then we write $\gamma_2 = \mathrm{card}(\Gamma_2)$, else $\gamma_2 = 0$. Then we compute

$$\gamma = \gamma_2 + \mathrm{card}\left(\bigcup_{\substack{3\leqslant k\leqslant m^+\\ k \text{ odd}}} \Gamma_k\right).$$

(1.3) For the singular points of order 1 or 2, $c \in \Gamma_2 \cup \Gamma_1$, we compute the principal parts of $g$:

$$g_c = \alpha_c(x-c)^{-2} + \beta_c(x-c)^{-1} + O(1),$$

if $c \in \Gamma'$, and

$$g_\infty = \alpha_\infty x^{-2} + \beta_\infty x^{-3} + O(x^{-4}),$$

for the point at infinity.

(1.4) We define the subset $L'$ (of all possible values for the degree of minimal polynomial) as $\{1\} \subset L'$ if $\gamma = \gamma_2$, $\{2\} \subset L'$ if $\gamma \geqslant 2$ and $\{4, 6, 12\} \subset L'$ if $m^+ \leqslant 2$.

(1.5) We have the three following mutually exclusive cases:

(1.5.1) If $m^+ > 2$, then $L = L'$.

(1.5.2) Define $\Delta_c = \sqrt{1+4\alpha_c}$. If $m^+ \leqslant 2$ and $\forall c \in \Gamma_1 \cup \Gamma_2$, $\Delta_c \in \mathbb{Q}$, then $L = L'$.

(1.5.3) If the cases (1.5.1) and (1.5.2) do not hold, then $L = L' - \{4, 6, 12\}$.

(1.6) If $L = \emptyset$, then equation (A.2) is nonintegrable with the Galois group $SL(2,\mathbb{C})$, else one writes $n$ for the minimum value in $L$.

For the second and third steps of the algorithm, we consider a fixed value of $n$.

*Step 2.*

(2.1) If $\infty$ has order 0, we write the set

$$E_\infty = \left\{0, \frac{h(n)}{n}, 2\frac{h(n)}{n}, 3\frac{h(n)}{n}, \ldots, n\frac{h(n)}{n}\right\}.$$

(2.2) If $c$ has order 1, then $E_c = \{h(n)\}$.

(2.3) If $n = 1$, for each $c$ of order 2, we define

$$E_c = \left\{\tfrac{1}{2}(1+\Delta_c), \tfrac{1}{2}(1-\Delta_c)\right\}.$$

(2.4) If $n \geqslant 2$, for each $c$ of order 2, we define

$$E_c = \mathbb{Z} \cap \left\{\frac{h(n)}{2}(1-\Delta_c) + \frac{h(n)}{n}k\Delta_c : k = 0, 1, \ldots, n\right\}.$$

(2.5) If $n = 1$, for each singular point of even order $2\nu$, with $\nu > 1$, we compute the numbers $\alpha_c$ and $\beta_c$ defined (up to a sign) by the following conditions:

(2.5.1) If $c \in \Gamma'$,

$$g_c = \left\{ \frac{\alpha_c}{(x-c)^\nu} + \sum_{i=2}^{\nu-1} \frac{\mu_{i,c}}{(x-c)^i} \right\}^2 + \frac{\beta_c}{(x-c)^{\nu+1}} + O\,(x-c)^{-\nu},$$

and we write

$$\sqrt{g_c} := \alpha_c\,(x-c)^{-\nu} + \sum_{i=2}^{\nu-1} \mu_{i,c}\,(x-c)^{-i}.$$

(2.5.2) If $c = \infty$,

$$g_\infty = \left\{ \alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i \right\}^2 - \beta_\infty x^{\nu-3} + O(x^{\nu-4}),$$

and we write

$$\sqrt{g_\infty} := \alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i.$$

Then for each $c$ as above, we compute

$$E_c = \left\{ \frac{1}{2}\left(\nu + \epsilon\frac{\beta_c}{\alpha_c}\right) : \epsilon = \pm 1 \right\},$$

and the sign function on $E_c$ is defined by

$$\operatorname{sign}\left(\frac{1}{2}\left(\nu + \epsilon\frac{\beta_c}{\alpha_c}\right)\right) = \epsilon,$$

being $+1$ if $\beta_c = 0$.

(2.6) If $n = 2$, for each $c$ of order $\nu$, with $\nu \geqslant 3$, we write $E_c = \{\nu\}$.

*Step 3.*

(3.1) For $n$ fixed, we try to obtain elements $\mathbf{e} = (e_c)_{c\in\Gamma}$ in the Cartesian product $\prod_{c\in\Gamma} E_c$, such that

(i) $d(\mathbf{e}) := n - \frac{n}{h(n)} \sum_{c\in\Gamma} e_c$ is a non-negative integer;

(ii) if $n = 2$ or $n = 6$, then $\mathbf{e}$ has an even number of elements which are odd integers;

(iii) when $n = 4$, then $\mathbf{e}$ has at least two elements not divisible by 3, and the sum of all elements not divisible by 3 is divisible by 3.

If no such set $\mathbf{e}$ is obtained, we select the next value in $L$ and repeat step 2, else $n$ is the maximum value in $L$ and the Galois group is $SL(2, \mathbb{C})$ (and equation (A.2) is nonintegrable).

(3.2) For each family $\mathbf{e}$ as above, we try to obtain a rational function $Q$ and a polynomial $P$, such that

(i)

$$Q = \frac{n}{h(n)} \sum_{c\in\Gamma'} \frac{e_c}{x-c} + \delta_{n1} \sum_{c\in\bigcup_{\nu>1}\Gamma_{2\nu}} \operatorname{sign}(e_c)\sqrt{g_c},$$

where $\delta_{n1}$ is the Kronecker delta.

(ii) $P$ is a polynomial of degree $d(\mathbf{e})$ and its coefficients are found as a solution of the (in general, overdetermined) system of equations

$$\begin{aligned} &P_{-1} = 0, \\ &P_{i-1} = -(P_i)' - QP_i - (n-i)(i+1)gP_{i+1}, \qquad n \geqslant i \geqslant 0, \\ &P_n = -P. \end{aligned}$$

If a pair $(P, Q)$ as above is found, then equation (A.2) is integrable and the Riccati equation (A.4) has an algebraic solution $v$ given by any root $v$ of the equation

$$f(v) = \sum_{i=0}^{n} \frac{P_i}{(n-i)!} v^i = 0.$$

If no pair as above is found, we take the next value in $L$ and go to step 2. If $n$ is the greatest value in $L$, then the Galois group of (A.2) is $SL(2, \mathbf{C})$ and the ODE is nonintegrable.

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